J. Fluid Mech. (1983), *vol.* 133, *pp.* 207–216
Printed in Great Britain

Wave evolution over a gradual slope with turbulent friction

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(Received 11 November 1982 and in revised form 14 March 1983)

The evolution of a weakly nonlinear, weakly dispersive gravity wave in water of depth d over a bottom of gradual slope δ and Chezy friction coefficient C_f is studied. It is found that an initially sinusoidal wave evolves into a periodic sequence of solitary waves with relative amplitude $a/d = \alpha_1 = 15\delta/4C_f$ if $\alpha_1 < \alpha_b$, where α_b is the relative amplitude above which breaking occurs. This prediction is supported by observations (Wells 1978) of the evolution of swell over mudflats.

1. Introduction

I consider here the evolution of a gravity wave of peak-to-trough amplitude a and length l in water of depth d over a bottom of slope δ on the following assumptions: (i) weak nonlinearity ($a \ll d$); (ii) weak dispersion ($d \ll l$); (iii) adiabatic variation ($\delta \ll d/l$); (iv) Chezy-type bottom friction,

$$\tau = C_f \rho \,|u|\, u, \tag{1.1}$$

where τ is the shear stress at the bottom, the constant C_f is Chezy's coefficient,† ρ is the density, and u is the particle velocity just outside of the boundary layer. Assumptions (i)–(iii) are characterized by the chain inequality

$$a \ll d \ll l \ll \frac{d}{\delta}. \tag{1.2a}$$

I also assume

$$\delta \ll C_f, \tag{1.2b}$$

which proves to be a necessary condition for a/d to remain small as $d \downarrow 0$, and provide for gradual refraction by regarding the wave as propagating in a virtual channel of gradually varying breadth b ($|\mathrm{d}b/\mathrm{d}x| \ll b/l$).

A similar investigation, also based on (i)–(iv), has been carried out by Shuto (1977). The present results are more compact, are useful over a wider parametric range, and provide explicit asymptotic approximations that permit more direct comparison with observation.

My interest in this problem was stimulated by the observations of Wells (1978) of wave evolution over shallow mudbanks on the coast of Surinam. Wells concluded that an initially sinusoidal wave (i.e. the dominant spectral component of incoming swell) evolves into a periodic sequence of solitary waves (cf. Munk 1949) with $a \propto d$. He suggested that this is at least partially a consequence of laminar friction, either in

† The Chezy coefficient actually depends on the amplitude and period of the wave motion, but the range of its variation is likely to be less than the uncertainty in its mean value in the present context. See Knight (1978) and Jonsson (1980) for information and additional references on C_f for oscillatory boundary layers.

the bottom boundary layer or in the mud, *qua* liquid, but both his own observations of the turbidity of the flow and (as I proceed to show) the asymptotic result $a \propto d$ suggest that the friction must be turbulent.

The analytical investigation of wave evolution with gradually changing depth goes back to Green (Lamb 1932, § 185), who predicted that long waves of sufficiently small amplitude evolve according to

$$a \propto d^{-\frac{1}{4}} \tag{1.3}$$

in the absence of dissipation. Boussinesq (1872) predicted that a solitary wave evolves according to

$$a \propto d^{-1}. \tag{1.4}$$

Both of these results may be inferred directly from conservation of energy on the assumption that dissipation and reflexion are negligible; both fail for sufficiently small d.

A shallow-water ($d \ll l$) sinusoidal wave of period T may be expected to remain sinusoidal in water of gradually varying depth if and only if both the *relative amplitude*

$$\alpha \equiv a/d \tag{1.5}$$

and the *Ursell parameter*

$$U \equiv \frac{aL}{d^2} \quad (L \equiv gT^2), \tag{1.6}$$

which is a measure of nonlinearity/dispersion,† remain small. If α does not remain small the wave may be expected to break; if α remains small but U does not the wave may be expected to evolve into a cnoidal wave and ultimately into a periodic sequence of solitary waves. It is evident from (1.3) and (1.4) that neither α nor U remain small over a shoaling bottom ($d \downarrow 0$) in the absence of dissipation. The hypothetical increase in amplitude due to shoaling is at least partially countered by friction, however, and wave evolution over a sufficiently gentle slope then is governed by a balance between the decay of wave energy and the power dissipated by bottom friction. I derive the corresponding energy-transport equation in §2 and apply it to sinusoidal, solitary and cnoidal waves in §§3–5. The results for a solitary wave (§4) have been previously reported (Miles 1983).

The results for sinusoidal and solitary waves over a uniform slope,

$$d = d_0 - \delta x \tag{1.7}$$

(x increases in the direction of wave propagation), are algebraically simple and may be summarized as follows. The relative amplitude of a sinusoidal wave increases/decreases monotonically (in the direction of decreasing depth) if $\alpha_0 \lessgtr \alpha_*$, where $\alpha_0 \equiv \alpha(0)$ and $\alpha_* = 15\pi\delta/8C_f$. The Ursell parameter U increases monotonically if $\alpha_0 < \frac{9}{5}\alpha_*$; if $\alpha_0 > \frac{9}{5}\alpha_*$, U initially decreases to a minimum and then increases monotonically to ∞ as $d/d_0 \downarrow 0$. The relative amplitude of a solitary wave increases/decreases monotonically if $\alpha_0 \lessgtr \alpha_1$, where

$$\alpha_1 = \frac{15}{4}\frac{\delta}{C_f}, \tag{1.8}$$

and is asymptotic to α_1:

$$a \sim \alpha_1 d \quad (d \downarrow 0). \tag{1.9}$$

† The wavelength is $l = cT = (dL)^{\frac{1}{2}}$, nonlinearity and dispersion are measured by a/d and d^2/l^2 respectively, and $U = al^2/d^3 = aL/d^2$.

The Ursell parameter for a cnoidal wave that is approximated by a sequence of solitary waves increases monotonically if $\alpha_0 < \frac{3}{2}\alpha_1$; if $\alpha_0 > \frac{3}{2}\alpha_1$, U initially decreases to a minimum and then increases monotonically. The asymptotic result (1.9) is valid for any shoaling bottom for which $d'(x) \to -\delta$ as $d \downarrow 0$ (see §4).

These results suggest that an initially sinusoidal wave for which α_0 and U_0 are small should evolve into a cnoidal wave and ultimately into a sequence of solitary waves with $a \sim \alpha_1 d$ as $d \downarrow 0$ over a sloping bottom for which $\alpha_1 < \alpha_b$, where $\alpha_b \approx 0.6$–0.8 is that value for α for which a solitary wave breaks (for references see Miles 1980); this conjecture is analytically confirmed in §5. If $\alpha_1 > \alpha_b$ the wave must be expected to break, although it could evolve into a sequence of solitary waves prior to breaking. Typical values of C_f are of the order of 10^{-2}, which suggests that breaking will precede the asymptotic limit $\alpha \sim \alpha_1$ unless δ is quite small (but a broken wave could reform and then evolve according to the present model). Wells (1978) reports $a \sim 0.23d$ on a slope of $\delta = 0.0005$, which provides at least qualitative confirmation of the present predictions; the corresponding value of C_f, inferred from (1.8), is 0.008.

It is instructive to inquire how the preceding results would be modified by the assumption of laminar, rather than turbulent, friction. I carry out the required calculation in the appendix and find that

$$a \sim 4.9 \times 10^4 \delta^4 \frac{g}{\nu^2} d^4 \quad (d \downarrow 0), \tag{1.10}$$

where ν is the kinematic viscosity. Wells reports values of ν between 0.02 and 275 for the Surinam mudbanks, which renders quantitative estimates from (1.10) somewhat uncertain; however, the qualitative estimate $a \propto d^4$ differs so markedly from the observed result $a \propto d$ as almost certainly to rule out laminar friction.

2. Energy-transport equation

Let $\eta(s, x)$ be the free-surface displacement of a wave in a channel of gradually varying breadth $b(x)$ and depth $d(x)$, where

$$s = \int \frac{\mathrm{d}x}{c} - t \tag{2.1}$$

is a characteristic coordinate,

$$c = (gd)^{\frac{1}{2}} \tag{2.2}$$

is the speed of an infinitesimal, shallow-water wave, and the conditions described in the first paragraph of §1 are assumed to hold. The wave energy then is $\rho g \eta^2$ per unit area, the energy flux is $\rho g \eta^2 bc$, the corresponding dissipation rate owing to bottom friction is $-\tau u b$, where τ is given by (1.1), and the particle velocity u is given by the shallow-water approximation (Lamb 1932, §169)

$$u = c\eta / d. \tag{2.3}$$

Averaging the energy and the dissipation rate over s on the assumption that η is periodic in s with period T, we obtain the energy-transport equation

$$\frac{\partial}{\partial x}(\rho g \langle \eta^2 \rangle bc) = -\langle \tau u \rangle b, \tag{2.4}$$

where

$$\langle f \rangle \equiv \frac{1}{T} \int_0^T f \, \mathrm{d}s. \tag{2.5}$$

Substituting (1.1), (2.2) and (2.3) into (2.4) and dividing through by $\rho g^{\frac{3}{2}}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(bd^{\frac{1}{2}}\langle\eta^2\rangle)+C_{\mathrm{f}}\,bd^{-\frac{3}{2}}\langle|\eta|^3\rangle = 0. \tag{2.6}$$

This last result also may be obtained by multiplying Shuto's (1977) equation (6) through by $2bc\eta$ and averaging the result over $\xi \equiv s$.

If η is aperiodic and vanishes at $s = \pm\infty$, (2.6) may be replaced by

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(bd^{\frac{1}{2}}\int_{-\infty}^{\infty}\eta^2\,\mathrm{d}s\right)+C_{\mathrm{f}}\,bd^{-\frac{3}{2}}\int_{-\infty}^{\infty}|\eta|^3\,\mathrm{d}s = 0. \tag{2.7}$$

It should be emphasized that the spatial intervals of the integrations with respect to s in (2.4)–(2.7) are, by assumption, small compared with the distance over which b and d exhibit appreciable changes. Moreover, the sidewalls of our channel are virtual and are to be interpreted as projections of rays on the free surface (cf. Ostrovsky 1976; Miles 1977). If the sidewalls are solid and have the same roughness as the bottom, as in a laboratory channel, C_{f} must be multiplied by $1+2d/b$ in (2.6) and (2.7).

We choose

$$b = \text{constant}, \quad d = d_0-\delta x \tag{2.8}$$

in the subsequent examples in order to simplify the results. The accommodation of other forms of b and d is straightforward.

3. Slowly varying sinusoidal wave

If both nonlinearity and dispersion are neglected, a sinusoidal wave of period T is described by the adiabatic approximation

$$\eta(s,x) = \tfrac{1}{2}a(x)\cos\frac{2\pi s}{T}. \tag{3.1}$$

(The definition of a as the peak-to-trough amplitude is consistent with the corresponding definitions for the solitary and cnoidal waves in §§4 and 5.) Substituting (3.1) into (2.6), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(a^2bd^{\frac{1}{2}})+\frac{4C_{\mathrm{f}}}{3\pi}a^3bd^{-\frac{3}{2}} = 0, \tag{3.2}$$

the integration of which from the reference point $x = 0$ with b and d given by (2.8) yields

$$a = a_0\left(\frac{d_0}{d}\right)^{\frac{1}{4}}\left[1+\left(\frac{8C_{\mathrm{f}}}{15\pi\delta}\right)\left(\frac{a_0}{d_0}\right)\left(\left(\frac{d_0}{d}\right)^{\frac{5}{4}}-1\right)\right]^{-1} \tag{3.3a}$$

$$\sim \alpha_* d \quad (d\downarrow 0), \tag{3.3b}$$

where $a_0 \equiv a(x_0)$ and

$$\alpha_* \equiv \tfrac{15}{8}\pi\frac{\delta}{C_{\mathrm{f}}}. \tag{3.4}$$

Introducing the relative amplitude $\alpha = a/d$, we obtain the alternative form

$$\frac{1}{\alpha} = \frac{1}{\alpha_0}\left(\frac{d}{d_0}\right)^{\frac{5}{4}}+\frac{1}{\alpha_*}\left[1-\left(\frac{d}{d_0}\right)^{\frac{5}{4}}\right]. \tag{3.5}$$

We remark that: α is a monotonically increasing/decreasing function of x if $\alpha_0 \lessgtr \alpha_*$; a has the maximum value

$$a_{\max} = \frac{1}{5}\left(\frac{\alpha_*}{\alpha_0}\right)\left(\frac{4\alpha_0}{\alpha_* - \alpha_0}\right)^{\frac{4}{5}} a_0 \quad \text{at} \quad \frac{d}{d_0} = \left(\frac{4\alpha_0}{\alpha_* - \alpha_0}\right)^{\frac{4}{5}} \tag{3.6a, b}$$

if $\alpha_* > 5\alpha_0$, but is a monotonically decreasing function of x (in $0 < x < d_0/\delta$) if $\alpha_* \leqslant 5\alpha_0$.

The results in this section are valid only if both the relative amplitude α and the Ursell parameter U remain small. Substituting (3.3a) into (1.6) and invoking (3.4), we obtain

$$U \equiv \frac{aL}{d^2} = U_0\left(\frac{d_0}{d}\right)^{\frac{9}{4}}\left[1 + \left(\frac{\alpha_0}{\alpha_*}\right)\left(\left(\frac{d_0}{d}\right)^{\frac{5}{4}} - 1\right)\right]^{-1} \tag{3.7a}$$

$$\sim \frac{\alpha_* L}{d} \quad (d \downarrow 0). \tag{3.7b}$$

It follows from (3.7a) that U is a monotonically increasing function of x if $\alpha_0 < \frac{9}{5}\alpha_*$; if $\alpha_0 > \frac{9}{5}\alpha_*$, U initially decreases to a minimum of

$$U_{\min} = \frac{9}{5}\left(\frac{9}{4}\right)^{\frac{4}{5}}\left(\frac{\alpha_*}{\alpha_0}\right)\left(\frac{\alpha_0 - \alpha_*}{\alpha_0}\right)^{\frac{4}{5}} U_0 \quad \text{at} \quad \left(\frac{d}{d_0}\right)^{\frac{5}{4}} = \frac{4}{9}\left(\frac{\alpha_0}{\alpha_0 - \alpha_*}\right), \tag{3.8a, b}$$

but then increases monotonically. It follows that the wave cannot remain sinusoidal even though $\alpha(x)$ remains small.

4. Slowly varying solitary wave

The adiabatic approximation for a solitary wave is given by (Miles 1979)

$$\eta(s, x) = a(x)\operatorname{sech}^2[\Omega(x)\{s - \sigma(x)\}], \tag{4.1}$$

where

$$\Omega(x) = \tfrac{1}{2}(3ga)^{\frac{1}{2}} d^{-1}, \quad \sigma(x) = \frac{1}{2}\int \frac{a}{d}\frac{\mathrm{d}x}{c}. \tag{4.2a, b}$$

Note that the introduction of $\sigma(x)$ in (4.1) is equivalent to the replacement of c by

$$\{g(d+a)\}^{\frac{1}{2}} \approx (gd)^{\frac{1}{2}}\left(1 + \frac{a}{2d}\right)$$

in (2.1).

Substituting (4.1) into (2.7), invoking (2.8), and carrying out the integrations, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(a^{\frac{3}{2}} b d^{\frac{3}{2}}) + \tfrac{4}{5} C_{\mathrm{f}}\, a^{\frac{5}{2}} b d^{-\frac{1}{2}} = 0, \tag{4.3}$$

$$a = a_0\left(\frac{d_0}{d}\right)\left[1 + \left(\frac{4}{15}\frac{C_{\mathrm{f}}}{\delta}\right)\left(\frac{a_0}{d_0}\right)\left(\frac{d_0^2}{d^2} - 1\right)\right]^{-1} \tag{4.4a}$$

$$\sim \alpha_1 d \quad (d \downarrow 0) \tag{4.4b}$$

$$\frac{1}{\alpha} = \frac{1}{\alpha_0}\left(\frac{d}{d_0}\right)^2 + \frac{1}{\alpha_1}\left(1 - \frac{d^2}{d_0^2}\right), \tag{4.5}$$

$$\alpha_1 \equiv \frac{15}{4}\frac{\delta}{C_{\mathrm{f}}} = \frac{4}{\pi}\alpha_* \tag{4.6}$$

and

$$U = U_0\left(\frac{d_0}{d}\right)^3\left[1+\frac{\alpha_0}{\alpha_1}\left(\frac{d_0^2}{d^2}-1\right)\right]^{-1} \tag{4.7a}$$

$$\sim \alpha_1\frac{L}{d} \quad (d\downarrow 0) \tag{4.7b}$$

as the counterparts of (3.2), (3.3a, b), (3.5), (3.4) and (3.7a, b). We remark that: α is a monotonically increasing/decreasing function of x if $\alpha_0 \lessgtr \alpha_1$; a has the maximum value

$$a_{\max} = \frac{1}{2}\left(\frac{\alpha_1}{\alpha_0}\right)\left(\frac{\alpha_1}{\alpha_0}-1\right)^{-\frac{1}{2}}a_0 \quad \text{at} \quad d = \left(\frac{\alpha_1}{\alpha_0}-1\right)^{-\frac{1}{2}}d_0 \tag{4.8}$$

if $\alpha_1 > 2\alpha_0$, but is a monotonically decreasing function of x (in $0 < x < d_0/\delta$) if $\alpha_1 \leqslant 2\alpha_0$; U is a monotonically increasing function of x if $\alpha_0 < \frac{3}{2}\alpha_1$; if $\alpha_0 > \frac{3}{2}\alpha_1$, U initially decreases to a minimum of

$$U_{\min} = \frac{3^{\frac{3}{2}}}{2}\left(\frac{\alpha_1}{\alpha_0}\right)\left(1-\frac{\alpha_1}{\alpha_0}\right)^{\frac{1}{2}}U_0 \quad \text{at} \quad \left(\frac{d}{d_0}\right)^2 = \frac{1}{3}\left(1-\frac{\alpha_1}{\alpha_0}\right)^{-1}, \tag{4.9a, b}$$

but ultimately increases monotonically.

We infer directly from the differential equation (4.3), which is singular at $d = 0$, that the asymptotic result $a \sim \alpha_1 d$, with α_1 given by (4.6), is valid for any shoaling bottom for which $b \to b_1 > 0$ and $d'(x) \to -\delta$ as $d\downarrow 0$; see (5.18) below. It is worth emphasizing that C_f may vary with x (owing to variations in both depth and bottom roughness) and that its value near $d = 0$ should be used in the calculation of α_1.

The approximation of a periodic wave by a sequence of solitary waves rests on the assumptions that α remains small and U remains large. A rough criterion for the approximation of the elliptic function cn $(u|m)$ by sech u is $U \gtrsim 70$, which corresponds to $0.99 \leqslant m < 1$; however, the variation of U implies significant dynamical effects for $U \lesssim 10^3$ (see below). Laboratory observations (for references see Miles 1980) suggest that (4.1) should provide a good approximation to a solitary wave if $\alpha \lesssim 0.5$ but that breaking should be expected for $\alpha > \alpha_b \approx 0.6$–$0.8$ (the experimental/theoretical value for breaking of a steady solitary wave is $\alpha_b = 0.6$–$0.7/0.83$).

5. Slowly varying cnoidal wave

The adiabatic approximation for a cnoidal wave of period $T \equiv (L/g)^{\frac{1}{2}}$ is given by (Miles 1979)

$$\eta(s,x) = a(x)\{\mathrm{cn}^2(2K\theta|m) - \langle \mathrm{cn}^2\rangle\}, \tag{5.1}$$

where

$$\frac{aL}{d^2} = \tfrac{16}{3}mK^2 \equiv U(m), \tag{5.2}$$

$$\theta = \frac{s-\sigma(x)}{T}, \quad \sigma = \frac{1}{2}\int\left[\frac{2-m-3(E/K)}{m}\right]\left(\frac{a}{d}\right)\frac{\mathrm{d}x}{c}, \tag{5.3a, b}$$

cn is an elliptic cosine of slowly varying modulus $m^{\frac{1}{2}}$ in the notation of Abramowitz & Stegun (1965), K and E are complete elliptic integrals of the first and second kind, U is a local Ursell parameter, and $\langle\ \rangle$ now signifies an average over the period of the elliptic functions; in particular,

$$\langle \mathrm{cn}^2\rangle \equiv \frac{1}{2K}\int_{-K}^{K}\mathrm{cn}^2(u|m)\,\mathrm{d}u = \frac{(E/K)-(1-m)}{m}. \tag{5.4}$$

We remark that (5.1) reduces to (3.1)/(4.1) in the limit $U\downarrow 0/\uparrow\infty$.

Substituting (5.1) into (2.6), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(IU^2bd^{\frac{9}{2}})+C_\mathrm{f}\,L^{-1}JU^3bd^{\frac{9}{2}}=0, \tag{5.5}$$

where

$$I=\langle \mathrm{cn}^4\rangle-\langle \mathrm{cn}^2\rangle^2, \tag{5.6}$$

$$J=\langle |\mathrm{cn}^2-\langle \mathrm{cn}^2\rangle|^3\rangle. \tag{5.7}$$

It is readily shown that (5.5) is equivalent to (3.2)/(4.3) in the limit $U\downarrow 0/\uparrow\infty$.

Turning to the uniform slope described by (2.8), we regard either m or U as the independent variable and

$$f\equiv\frac{\alpha(x)}{\alpha_1}=\left(\frac{4}{15}\frac{C_\mathrm{f}}{\delta}\right)\frac{a(x)}{d(x)} \tag{5.8}$$

as the dependent variable. Combining (5.2) and (5.8) to obtain $d=(15\delta L/4C_\mathrm{f})\,(f/U)$ and substituting this result, together with (2.8), into (5.5), we obtain

$$(f-f_1)\,U\frac{\mathrm{d}f}{\mathrm{d}U}=f(f-f_2), \tag{5.9}$$

where

$$f_1=\frac{6I}{5J},\quad f_2=\frac{2I}{3J}\left(1-\frac{2}{5}\frac{\mathrm{d}\ln I}{\mathrm{d}\ln U}\right). \tag{5.10a, b}$$

The parameters I, U, $\mathrm{d}I/\mathrm{d}m$ and $dU/\mathrm{d}m$ may be calculated from (5.2) and (5.6) with the aid of formulas in Byrd & Friedman (1954) to obtain

$$I=\tfrac{1}{3}m^{-2}\{2(2-m)\,\mathscr{E}-3\mathscr{E}^2-(1-m)\}, \tag{5.11}$$

$$\frac{\mathrm{d}\ln U}{\mathrm{d}m}=m^{-1}(1-m)^{-1}\mathscr{E}, \tag{5.12}$$

$$1-\frac{2}{5}\frac{\mathrm{d}\ln I}{\mathrm{d}\ln U}=\frac{3}{5}\left[\frac{4(2-m)\,\mathscr{E}-7\mathscr{E}^2-(1-m)}{2(2-m)\,\mathscr{E}-3\mathscr{E}^2-(1-m)}\right], \tag{5.13}$$

where

$$\mathscr{E}=E(m)/K(m). \tag{5.14}$$

Numerical investigation reveals that the integral J varies by only a few percent from $1/6\pi$ (its value at $m=0$) for $0\leqslant m\leqslant 0.99$ and is given by (5.17) below with an error of less than 1 % for $m\geqslant 0.99$.

The limit $m\uparrow 1$ is logarithmically slow, in consequence of which it is expedient to neglect $1-m$ except in its logarithm if $1-m\ll 1$ and invoke the asymptotic approximations

$$U\sim\tfrac{16}{3}K^2\sim\tfrac{4}{3}\ln^2\frac{16}{1-m}\quad(m\uparrow 1), \tag{5.15}$$

$$f_1\sim\tfrac{4}{5}(JK)^{-1}\,(1-\tfrac{3}{2}K^{-1}),\quad f_2\sim\tfrac{8}{15}(JK)^{-1}\,(1-\tfrac{7}{4}K^{-1}), \tag{5.16a, b}$$

$$J\sim\tfrac{16}{15}K^{-1}(1-\tfrac{13}{4}K^{-1}+\tfrac{33}{8}K^{-2})\,(1-K^{-1})^{\frac{1}{2}}-\tfrac{8}{15}K^{-1}+2K^{-2}-2K^{-3}$$
$$-2K^{-4}\ln\{K^{\frac{1}{2}}+(K-1)^{\frac{1}{2}}\}. \tag{5.17}$$

The differential equation (5.9), which is of Abel's type, has four singularities in a (U,f)-plane (it is analytically convenient to regard $1/U$ and f as the phase-plane variables near $U=\infty$):

(i) a node at $U=0$ and $f=0$, in the neighbourhood of which $f\propto U^{\frac{5}{9}}$, or, equivalently, $a\propto d^{-\frac{1}{4}}$ (cf. (1.3)); (ii) a saddle point at $U=0$ and $f=f_2=\tfrac{1}{2}\pi$; (iii) a

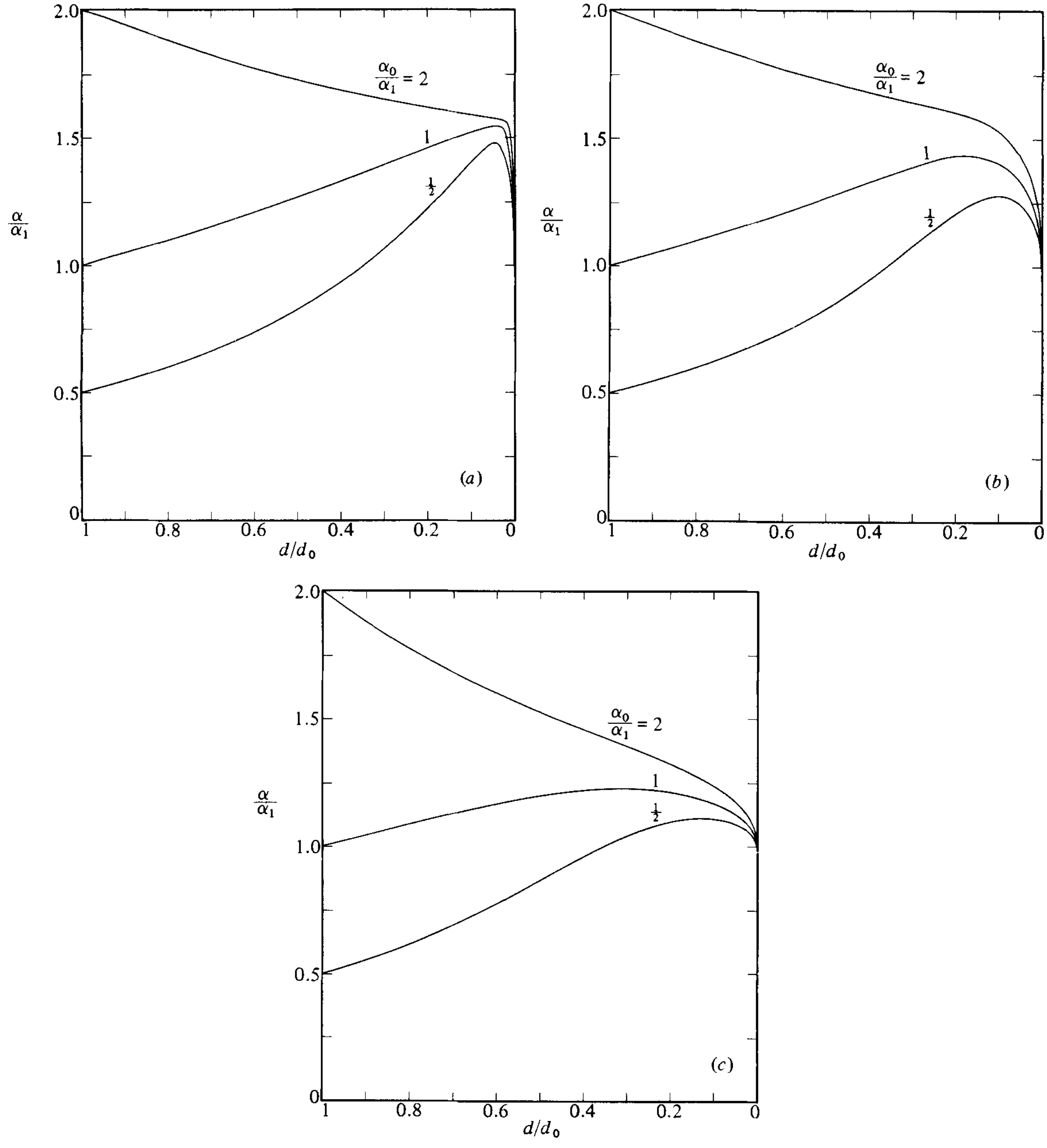

FIGURE 1. α/α_1 *vs.* d/d_0 for (*a*) $U_0 = 1$, (*b*) $U_0 = 10$, (*c*) $U_0 = 100$.

saddle point at $U = \infty$ and $f = 0$; (iv) a node at $U = \infty$ and $f = f_2 = 1$, in the neighbourhood of which $f - 1 \propto U^{-2}$, or, equivalently (cf. (4.7*b*)),

$$a \sim \alpha_1 d\{1 + O(U^{-2})\} = \alpha_1 d\left\{1 + O\left(\frac{d^2}{L^2}\right)\right\} \quad (U \uparrow \infty). \tag{5.18}$$

The integration may be carried out numerically, starting from the point $f_0 = \alpha_0/\alpha_1$ and $U_0 = a_0 L/d_0^2$ and integrating toward the node at $f = 1$ and $U = \infty$ to obtain a

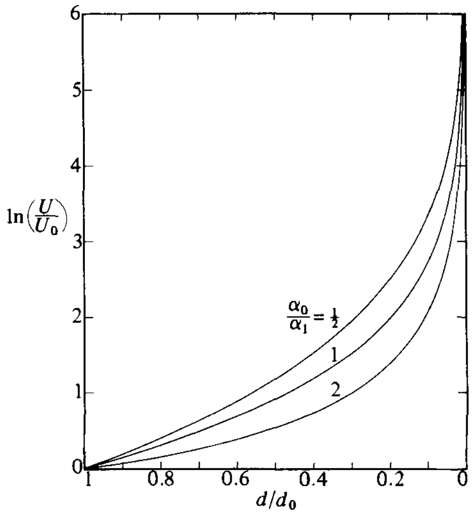

FIGURE 2. Plots of ln (U/U_0) *vs.* d/d_0 for $U_0 = 1$. The corresponding results for $U_0 = 10$ are almost coincident with, and those for $U_0 = 100$ are close to, those for $U_0 = 1$.

two-parameter (f_0 and U_0) family of solutions. The local depth, as determined from (5.2) and (5.8), is given by

$$\frac{d}{d_0} = \left(\frac{U_0}{U}\right)\left(\frac{f}{f_0}\right). \tag{5.19}$$

The results are plotted in figures 1 and 2.† We remark that, in contrast to the limiting results for sinusoidal and cnoidal waves (for which α either increases or decreases monotonically), α increases to a peak and then decreases to α_1 if $\alpha_0 < \alpha_1$.

This work was supported in part by the Physical Oceanography Division, National Science Foundation, NSF Grant OCE77-24005, and by a contract with the Office of Naval Research.

Appendix. Laminar friction

If the boundary layer is assumed to be laminar, rather than turbulent, (2.7) must be replaced by (cf. Miles 1976*a*)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(bc\int_{-\infty}^{\infty}\eta^2\,\mathrm{d}x\right)+\frac{1}{2}\frac{b}{d}\left(\frac{\nu}{\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\eta(s,x)\,\eta_s(s-\sigma,x)\frac{\operatorname{sgn}\sigma}{|\sigma|^{\frac{1}{2}}}\,\mathrm{d}\sigma = 0, \tag{A 1}$$

where ν is the kinematic viscosity. A computationally simpler form, obtained by introducing the Fourier transform

$$N(\omega,x) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega s}\,\eta(s,x)\,\mathrm{d}s \tag{A 2}$$

† The numerical integration of (5.9) was carried out with the independent variable m and the approximation $J = 0.05305$ for $0 < m < 0.99$, and with the independent variable U and the asymptotic approximations (5.15)–(5.17) for $m > 0.99$ ($U > 72$).

and invoking the Parseval and convolution theorems for the integrals in (A 1), is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(bc\int_{-\infty}^{\infty}|N|^2\,\mathrm{d}\omega\right)+\frac{b}{d}\left(\frac{\nu}{2}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}|\omega|^{\frac{1}{2}}|N|^2\,\mathrm{d}\omega=0. \tag{A 3}$$

Substituting (4.1) into (A 2) and the result into (A 3), we obtain (for details cf. Miles 1976*b*)

$$\frac{\mathrm{d}}{\mathrm{d}x}(a^{\frac{3}{2}}bd^{\frac{3}{2}})+3Cg^{-\frac{1}{4}}(\tfrac{1}{2}\nu)^{\frac{1}{2}}a^{\frac{7}{4}}bd^{-\frac{1}{2}}=0, \tag{A 4}$$

where $C = 0.2372\ldots$, and

$$a=a_0\frac{d_0}{d}\left[1+\frac{4}{5}\frac{C}{\delta}\left(\frac{\nu}{8c_0d_0}\right)^{\frac{1}{2}}\left(\frac{a_0}{d_0}\right)^{\frac{1}{4}}\left(\frac{d_0^{\frac{5}{4}}}{d^{\frac{5}{4}}}-1\right)\right]^{-4}. \tag{A 5a}$$

$$\sim 4.9\times10^4\delta^4(g/\nu^2)\,d^4\quad\left(\frac{d}{d_0}\downarrow 0\right) \tag{A 5b}$$

as the counterparts of (4.3) and (4.4*a*, *b*). It may be inferred directly from (A 4) that the asymptotic result (A 5*b*) holds for any shoaling bottom for which $b \rightarrow b_1 > 0$ and $d'(x) \rightarrow -\delta$ as $d \downarrow 0$.

Sidewall and free-surface-contaminant boundary layers may be accommodated by introducing the factor $1+C+2d/b$ in the last term in each of (A 1), (A 3) and (A 4), and in the second term within the brackets in (A 5*a*), where C is a surface-contamination coefficient that may be approximated by 1 for tap water.

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